Weak thermal vortex rings

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A similarity solution is obtained up to the first order in an effective Rayleigh number for the behaviour of very weak thermal vortex rings produced by the rapid release of heat at one point of a large region of fluid.

The laminar pattern of flow is similar to that in ordinary vortex rings, but the temperature decreases outwards from the centre in all directions with some asymmetry about the horizontal plane through the centre, and there is no accumulation of heat into the vortex ring. The vortex propagates slowly in relation to its rate of growth, and the process is dominated by viscous and thermal diffusion.

Introduction

The rapid release of a quantity of heat from a compact isolated source into a large region of fluid at rest will set in motion a small volume of heated fluid. As it rises, this buoyant fluid will grow in volume through conduction of heat and viscous diffusion of momentum outwards. Since the heated fluid is displaced upwards by ambient fluid from approximately its own level and in turn displaces ambient fluid from its path, the flow must have the general pattern of a vortex ring, although this may sometimes be obscured from the observer by an opaque envelope of marked fluid or by turbulence in and around the core.

The behaviour of buoyant vortices formed by the release of buoyant fluid from rest can be characterized by a non-dimensional parameter which is proportional to the initial release of buoyancy and which plays a part corresponding to that of the Rayleigh number in problems where a length and a temperature difference are specified; it will be convenient to refer to it in this case also as a Rayleigh number. By analogy with the buoyant plumes that rise from steady sources of heat or other sources of pure positive buoyancy (positive in the sense that upwards motion is induced) it may be expected that for small over-all Rayleigh numbers the flow will be laminar in and around the corresponding vortex rings while for large Rayleigh numbers it will be turbulent. But in fact there is an additional strong source of stability in buoyant vortex rings which have light cores because the reduced density of fluid in the ring produces a stable stratification in the field of centrifugal force due to rotation about the core. Thus thermal vortex rings of normal type should have a stable pattern of laminar flow over an appreciable range of Rayleigh numbers, which will certainly include the range of small values that is to be considered here. Indeed there is evidence to show that, when sufficient time is available, initially turbulent regions of rising buoyant fluid will form into relatively well-ordered vortex rings (see, for example, Turner 1957).

When the Rayleigh number is negative, vortex rings have a heavy core and their behaviour shows a marked contrast. The rotating core is now inherently unstable to disturbances and will break up unless the Rayleigh number is small; moreover, vortex rings will not so readily grow from less ordered motions, at any rate until density differences have been reduced to a low level. This type of behaviour may be illustrated by releasing drops of dyed salt solution just below the surface of a beaker of water.

It may be noted that this difference in behaviour according as the Rayleigh number is positive or negative must be borne in mind when model experiments are designed, unless all density variations are kept very small. A further factor is that the behaviour of the accelerating surface of a volume of buoyant fluid will be different according as the fluid moving forwards is heavier or lighter than its surroundings (cf. Taylor 1950).

The analysis of buoyant vortex rings which follows will be valid only for small values of the Rayleigh number, in which case a similarity solution can be found as a power series in the Rayleigh number for the momentum and energy equations including the effects of viscosity and thermal conductivity. Under these circumstances the motion may be expected to remain laminar for a considerable time in the absence of grave disturbances in the ambient fluid, and provided that the Rayleigh number is positive. Hence the solution will represent, for suitable initial conditions, the behaviour of weak thermal vortex rings generated from rest by the rapid release of a small quantity of heat in the close neighbourhood of one point of an extensive region of fluid. The discharge of a thermal vortex ring with given initial circulation can be investigated by using appropriately modified initial conditions, but will not be included here. The results given below may be regarded as providing an asymptotic solution for weak buoyant vortex rings; an approximate treatment has already been given by Turner (1957) for stronger buoyant vortices.

Formulation

The idealized problem is the ascent, through a uniform environment of incompressible fluid, of the vortex ring generated from an instantaneous point source of heat. The flow will start impulsively from the virtual source, and for small values of the Rayleigh number will remain laminar over an appreciable distance so that it is appropriate to seek a similarity type solution.

The motion is symmetrical about a vertical axis and it seems natural to use cylindrical polar co-ordinates for reference; in fact it is more convenient to use spherical polar co-ordinates (r, θ, ϕ) with origin at the point of release of heat and the axis $\theta = 0$ directed vertically upwards, because the analysis is concerned with finding patterns of velocity and temperature which in the similarity solution do not vary with time. The velocity components can be taken as $(u, v, 0)$ since the system is independent of ϕ; the time t will be measured from the instant of release. If it is now assumed: (i) that variations in density due to temperature changes are so small that they need only be taken into account in the buoyancy term; (ii) that the effects of dissipation and the pressure term in the energy equation are negligible; and (iii) that the kinematic viscosity ν and the thermometric conductivity κ may be taken as constants; then the equations that represent the convective flow are

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r}+\frac{v}{r}\frac{\partial u}{\partial \theta}-\frac{v^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r}+\nu\left(\nabla_1^2 u-\frac{2u}{r^2}-\frac{2v\cot\theta}{r^2}-\frac{2}{r^2}\frac{\partial v}{\partial \theta}\right)+\beta g(T-T_0)\cos\theta, \tag{1}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial r}+\frac{v}{r}\frac{\partial v}{\partial \theta}+\frac{uv}{r} = -\frac{1}{\rho r}\frac{\partial p}{\partial \theta}+\nu\left(\nabla_1^2 v-\frac{v}{r^2\sin^2\theta}+\frac{2}{r^2}\frac{\partial u}{\partial \theta}\right)-\beta g(T-T_0)\sin\theta, \tag{2}$$

$$\frac{\partial}{\partial r}(r^2 u\sin\theta)+\frac{\partial}{\partial \theta}(rv\sin\theta) = 0, \tag{3}$$

$$\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial r}+\frac{v}{r}\frac{\partial T}{\partial \theta} = \kappa\nabla_1^2 T, \tag{4}$$

where ρ is the fluid density, p the pressure excess over hydrostatic, T the temperature in the vortex ring and T_0 the uniform temperature of the ambient fluid, β the coefficient of expansion, and

$$\nabla_1^2 \equiv \frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\sin\theta\frac{\partial}{\partial \theta}.$$

A stream function ψ can be introduced, with

$$u = \frac{1}{r^2\sin\theta}\frac{\partial \psi}{\partial \theta},\quad v = -\frac{1}{r\sin\theta}\frac{\partial \psi}{\partial r}.$$

If p is now eliminated between (1) and (2), and (4) is rewritten in terms of the stream function,

$$\frac{\partial Z}{\partial t}+\frac{\partial(Z/r^2\sin\theta,\psi)}{\partial(r,\theta)} = \nu\left(\nabla_1^2 Z-\frac{2}{r}\frac{\partial Z}{\partial r}-\frac{Z}{r^2\sin^2\theta}\right)+\beta g\left(\frac{\partial T}{\partial \theta}\cos\theta+\frac{\partial T}{\partial r}r\sin\theta\right), \tag{5}$$

$$\frac{\partial T}{\partial t}+\frac{1}{r^2\sin\theta}\frac{\partial(T,\psi)}{\partial(r,\theta)} = \kappa\nabla_1^2 T, \tag{6}$$

where the vorticity has components $(0, 0, -Z/r)$, and

$$Z = \frac{\partial u}{\partial \theta}-\frac{\partial(rv)}{\partial r} = \operatorname{cosec}\theta\left(\frac{\partial^2\psi}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2\psi}{\partial \theta^2}-\frac{\cot\theta}{r^2}\frac{\partial \psi}{\partial \theta}\right).$$

The initial conditions are that u and v are zero and $T = T_0$ at $t = 0$ for all points of the field except $r = 0$, where there is a singular point; for $t > 0$ the field is free from singularities. The boundary conditions for $t > 0$ are

$$\left.\begin{aligned} &u, v = 0 \quad \text{and} \quad T = T_0 \quad \text{at} \quad r = \infty;\\ &\frac{1}{r}\frac{\partial u}{\partial \theta},\ v,\ \text{and}\ \ \frac{1}{r}\frac{\partial T}{\partial \theta} = 0 \quad \text{at} \quad \theta = 0, \pi.\end{aligned}\right\} \tag{7}$$

The excess of heat in the vortex ring over that in the same volume of ambient fluid remains constant and is equal to the amount Q of heat released initially,

$$2\pi\int_0^\infty\int_0^\pi (T-T_0)\,r^2\sin\theta\,d\theta\,dr = \frac{Q}{\rho c} = \frac{F}{\beta g},$$

where ρF is the total initial release of buoyancy.

If a similarity solution of (5) and (6) exists it must be such as to relate r and t in the group $rt^{-\frac{1}{2}}$ if t is not to appear explicitly in the transformed equations. Using the same criterion, it may be shown that (5) and (6) are reduced to non-dimensional form by the transformations

$$\eta = r/(2\kappa t)^{\frac{1}{2}},$$
$$\psi = \kappa(2\kappa t)^{\frac{1}{2}} f(\eta, \theta),$$
$$\beta(T - T_0) = \frac{1}{g} F(2\kappa t)^{-\frac{3}{2}} h(\eta, \theta).$$

Also put $$\zeta = (2t/\kappa)^{\frac{1}{2}} Z = \operatorname{cosec}\theta\left(\frac{\partial^2 f}{\partial \eta^2} + \frac{1}{\eta^2}\frac{\partial^2 f}{\partial \theta^2} - \frac{\cot\theta}{\eta^2}\frac{\partial f}{\partial \theta}\right).$$

The reduced equations are

$$-\left(\zeta + \eta\frac{\partial \zeta}{\partial \eta}\right) + \frac{\partial(\zeta/\eta^2 \sin\theta, f)}{\partial(\eta, \theta)}$$
$$= \sigma\left(\nabla^2_{\eta,\theta} - \frac{2}{\eta}\frac{\partial}{\partial \eta} - \frac{1}{\eta^2\sin^2\theta}\right)\zeta + \sigma A\left(\frac{\partial h}{\partial \theta}\cos\theta + \frac{\partial h}{\partial \eta}\eta\sin\theta\right), \tag{8}$$

$$-\left(3h + \eta\frac{\partial h}{\partial \eta}\right) + \frac{1}{\eta^2\sin\theta}\frac{\partial(h, f)}{\partial(\eta, \theta)} = \nabla^2_{\eta,\theta} h. \tag{9}$$

The transformed boundary conditions are:

$$\left.\begin{aligned} &\frac{1}{\eta^2\sin\theta}\frac{\partial f}{\partial \theta} = 0, \quad \frac{1}{\eta\sin\theta}\frac{\partial f}{\partial \eta} = 0, \quad h = 0, \quad \text{at} \quad \eta = \infty; \\ &\frac{1}{\eta}\frac{\partial}{\partial \theta}\frac{1}{\eta^2\sin\theta}\frac{\partial f}{\partial \theta} = 0, \quad \frac{1}{\eta\sin\theta}\frac{\partial f}{\partial \eta} = 0, \quad \frac{1}{\eta}\frac{\partial h}{\partial \theta} = 0, \quad \text{at} \quad \theta = 0, \pi; \end{aligned}\right\} \tag{10}$$

and the integral condition reduces to

$$2\pi \int_0^\infty \int_0^\pi \eta^2 \sin\theta\, h(\eta, \theta)\, d\theta\, d\eta = 1. \tag{11}$$

The non-dimensional parameter $A = \beta g Q/\rho c \kappa \nu = F/\kappa\nu$ plays the same part in this problem as that of the Rayleigh number for cases in which a length and a temperature difference are specified, and it will also be referred to as a Rayleigh number here. For small values of A the flow in the thermal vortex ring should certainly remain laminar for a considerable time, and as the exact solution of (5) and (6) will be difficult, this suggests that a solution should be obtained as a power series expansion in the Rayleigh number A. Hence assume the expansions

$$f = Af_1 + A^2 f_2 + \ldots, \tag{12}$$

$$h = h_0 + Ah_1 + A^2 h_2 + \ldots, \tag{13}$$

where $f_i = f_i(\eta, \theta)$ and $h_i = h_i(\eta, \theta)$. The term $f_0(\eta, \theta)$ is constant, since there is no motion when $A = 0$ (i.e. $\partial f/\partial \eta = 0$, $\partial f/\partial \theta = 0$ then), and this constant can be absorbed into f. Further, the value of $h_0(\eta, \theta)$ can be found immediately from the theory of conduction of heat in solids; for when $A = 0$ (provided that κ is finite and $Q \neq 0$) the heat spreads as by conduction in a uniform solid from an instantaneous point source. For such a case the solution given by Carslaw & Jaeger (1947) can be written in the present notation as

$$h_0(\eta,\theta) = (2\pi)^{-\frac{3}{2}}\, e^{-\frac{1}{2}\eta^2}. \tag{14}$$

The other coefficient functions $f_i(\eta,\theta)$ and $h_i(\eta,\theta)$ can be found by substituting the expansions (12) and (13) into equations (8) and (9) and equating to zero the coefficients of powers of A in the two relations to give two sets of linear partial differential equations:

$$-\left(\zeta_1+\eta\frac{\partial\zeta_1}{\partial\eta}\right) = \sigma\left(\nabla_1^2-\frac{2}{\eta}\frac{\partial}{\partial\eta}-\frac{1}{\eta^2\sin^2\theta}\right)\zeta_1+\sigma\left(\cos\theta\frac{\partial h_0}{\partial\theta}+\eta\sin\theta\frac{\partial h_0}{\partial\eta}\right), \tag{15a}$$

$$-\left(\zeta_2+\eta\frac{\partial\zeta_2}{\partial\eta}\right)+\frac{\partial(\zeta_1/\eta^2\sin\theta,f_1)}{\partial(\eta,\theta)} = \sigma\left(\nabla_1^2-\frac{2}{\eta}\frac{\partial}{\partial\eta}-\frac{1}{\eta^2\sin^2\theta}\right)\zeta_2 +\sigma\left(\cos\theta\frac{\partial h_1}{\partial\theta}+\eta\sin\theta\frac{\partial h_1}{\partial\eta}\right), \tag{15b}$$

$$\dots\quad\dots\quad\dots\quad\dots\quad\dots\quad\dots\quad\dots\quad\dots;$$

and

$$-\left(3h_0+\eta\frac{\partial h_0}{\partial\eta}\right) = \nabla_1^2 h_0, \tag{16a}$$

$$-\left(3h_1+\eta\frac{\partial h_1}{\partial\eta}\right)+\frac{1}{\eta^2\sin\theta}\frac{\partial(h_0,f_1)}{\partial(\eta,\theta)} = \nabla_1^2 h_1, \tag{16b}$$

$$-\left(3h_2+\eta\frac{\partial h_2}{\partial\eta}\right)+\frac{1}{\eta^2\sin\theta}\left\{\frac{\partial(h_0,f_2)}{\partial(\eta,\theta)}+\frac{\partial(h_1,f_1)}{\partial(\eta,\theta)}\right\} = \nabla_1^2 h_2, \tag{16c}$$

$$\dots\quad\dots\quad\dots\quad\dots\quad\dots\quad\dots\quad\dots\quad\dots;$$

where in equations (15) and (16) ζ has been taken in the form

$$\zeta = A\zeta_1(\eta,\theta)+A^2\zeta_2(\eta,\theta)+\dots,$$

so that

$$\zeta_i(\eta,\theta) = \operatorname{cosec}\theta\left(\frac{\partial^2}{\partial\eta^2}+\frac{1}{\eta^2}\frac{\partial^2}{\partial\theta^2}-\frac{\cot\theta}{\eta^2}\frac{\partial}{\partial\theta}\right)f_i,$$

and ∇_1^2 has been written in place of $\nabla_{\eta,\theta}^2$. The solution to equation (16a) satisfying the appropriate boundary conditions is known already; hence a solution of (15a) and (16b) will provide a first approximation to the behaviour of thermal vortex rings, that is a solution to the first order in the Rayleigh number.

The analysis simplifies a good deal for the particular case $\sigma = 1$, and as this is sufficient to give a good idea of the behaviour of thermal vortex rings in gases and to demonstrate the important properties of these rings, only this case will be pursued.

As a result of the boundary conditions (10), which do not depend on A, the following conditions are imposed on the f_i and h_i:

$$\left.\begin{aligned}&\frac{1}{\eta^2}\frac{\partial f_i}{\partial\theta}=0,\quad \frac{1}{\eta}\frac{\partial f_i}{\partial\eta}=0,\quad h_i=0\quad\text{at}\quad \eta=\infty;\\ &\frac{1}{\eta}\frac{\partial}{\partial\theta}\frac{1}{\eta^2\sin\theta}\frac{\partial f_i}{\partial\theta}=0,\quad \frac{1}{\eta\sin\theta}\frac{\partial f_i}{\partial\eta}=0,\quad \frac{1}{\eta}\frac{\partial h_i}{\partial\theta}=0\quad\text{at}\quad\theta=0,\pi;\end{aligned}\right\} \tag{17}$$

and from the integral condition (11),

$$2\pi \int_0^\infty \int_0^\pi \eta^2 \sin\theta \, h_0(\eta,\theta)\, d\theta\, d\eta = 1,$$

and

$$\int_0^\infty \int_0^\pi \eta^2 \sin\theta \, h_i(\eta,\theta)\, d\theta\, d\eta = 0 \quad \text{for} \quad i \geqslant 1.$$

Solution

The first approximation to the flow field. The vorticity variable ζ_1 satisfies the differential equation

$$-\left(\zeta_1 + \eta \frac{\partial \zeta_1}{\partial \eta}\right) = \left(\frac{1}{\eta^2}\frac{\partial}{\partial \eta}\eta^2 \frac{\partial}{\partial \eta} + \frac{1}{\eta^2 \sin\theta}\frac{\partial}{\partial \theta}\sin\theta \frac{\partial}{\partial \theta} - \frac{2}{\eta}\frac{\partial}{\partial \eta} - \frac{1}{\eta^2 \sin^2\theta}\right)\zeta_1$$
$$+ (2\pi)^{-\frac{3}{2}} \eta \sin\theta \frac{\partial}{\partial \eta}(e^{-\frac{1}{2}\eta^2}), \quad (18)$$

where the Prandtl number has been given the value $\sigma = 1$. No boundary conditions for ζ_1 have been stated, but those on the related stream function variable $f_1(\eta,\theta)$ are given in (17).

If ζ_1 is assumed to be of form $\zeta_1 = z(\eta)\sin\theta$, by substitution in (18) the dependent variable $z(\eta)$ must satisfy the ordinary differential equation

$$\frac{d^2 z}{d\eta^2} + \eta \frac{dz}{d\eta} + \left(1 - \frac{2}{\eta^2}\right) z = (2\pi)^{-\frac{3}{2}} \eta^2 e^{-\frac{1}{2}\eta^2}. \quad (19)$$

This has the general solution

$$z = \frac{1}{\eta}\left\{c_1 \int_0^\eta t^2 e^{-\frac{1}{2}t^2} dt + c_2\right\} - \tfrac{1}{2}(2\pi)^{-\frac{3}{2}} \eta^2 e^{-\frac{1}{2}\eta^2},$$

involving two arbitrary constants c_1 and c_2. Although boundary conditions on ζ_1 have not been stated, it is clear that $c_2 = 0$ since, as $\eta \to 0$, z cannot diverge; however, it seems probable that at great distances from the point of release of heat ζ will decrease exponentially, whereas the term in c_1 behaves as η^{-1} and hence $c_1 = 0$. It will be shown at the next stage that as a result of taking

$$\zeta_1 = -\tfrac{1}{2}(2\pi)^{-\frac{3}{2}} \eta^2 e^{-\frac{1}{2}\eta^2} \sin\theta, \quad (20)$$

the resultant solution for f_1 satisfies all the appropriate boundary conditions from set (17), and this confirms the value $c_1 = 0$.

The stream function $f_1(\eta,\theta)$ satisfies the equation

$$\frac{\partial^2 f_1}{\partial \eta^2} + \frac{1}{\eta^2}\frac{\partial^2 f_1}{\partial \theta^2} - \frac{\cot\theta}{\eta^2}\frac{\partial f_1}{\partial \theta} = \zeta_1 \sin\theta = -\tfrac{1}{2}(2\pi)^{-\frac{3}{2}} \eta^2 \sin^2\theta \, e^{-\frac{1}{2}\eta^2}. \quad (21)$$

In this case the independent variable can be separated if $f_1(\eta,\theta)$ is supposed to be of form $f_1(\eta,\theta) = y(\eta)\sin^2\theta$, where

$$\frac{d^2 y}{d\eta^2} - \frac{2}{\eta^2} y = -\tfrac{1}{2}(2\pi)^{-\frac{3}{2}} \eta^2 e^{-\frac{1}{2}\eta^2}. \quad (22)$$

Equation (22) has the general solution

$$y = \tfrac{1}{2}(2\pi)^{-\frac{3}{2}}\left\{c_3\eta^2 + \frac{c_4}{\eta} - e^{-\frac{1}{2}\eta^2} + \frac{1}{\eta}\int_0^\eta e^{-\frac{1}{2}t^2}\,dt\right\},$$

where the arbitrary constants c_3 and c_4 can here be found from boundary conditions (17) as $c_3 = 0$ and $c_4 = 0$. It follows that the stream function variable

$$\left.\begin{aligned} f_1 &= \tfrac{1}{2}(2\pi)^{-\frac{3}{2}}\left\{\frac{1}{\eta}\int_0^\eta e^{-\frac{1}{2}t^2}\,dt - e^{-\frac{1}{2}\eta^2}\right\}\sin^2\theta,\\ &= \tfrac{1}{2}(2\pi)^{-\frac{3}{2}}\left\{\sqrt{\frac{\pi}{2}}\,\frac{\operatorname{erf}(\eta/\sqrt{2})}{\eta} - e^{-\frac{1}{2}\eta^2}\right\}\sin^2\theta,\end{aligned}\right\} \qquad (23)$$

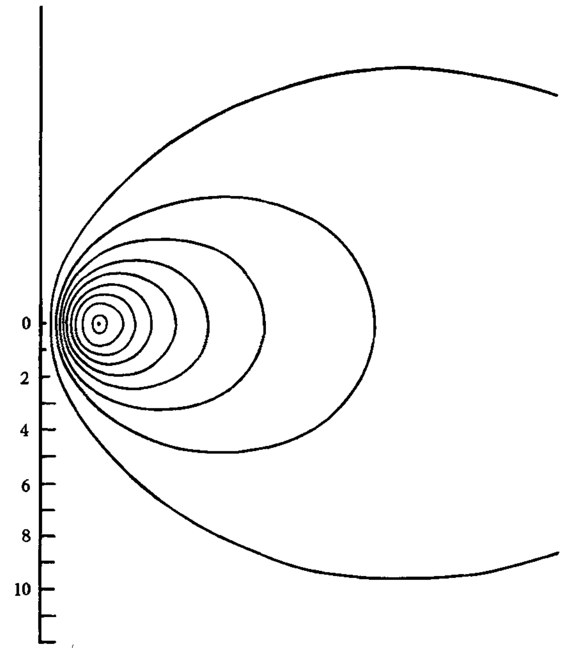

FIGURE 1. The streamlines for the first approximation to the flow in a weak thermal vortex ring. Curves are shown for equal increments of the stream function. Only the right-hand half of the field of flow is shown, and the scale for the non-dimensional length η is shown on the dividing streamline.

satisfies equation (18) written fully in terms of f_1, and also satisfies appropriate boundary conditions from the set (17), so that it gives the first approximation to the flow field. The corresponding contributions to the components of the velocity field are

$$\frac{A}{8\pi}\sqrt{\frac{\kappa}{\pi t}}\left\{\frac{2\cos\theta}{\eta^3}\left[\int_0^\eta e^{-\frac{1}{2}t^2}\,dt - \eta\, e^{-\frac{1}{2}\eta^2}\right],\quad \frac{\sin\theta}{\eta^3}\left[\int_0^\eta e^{-\frac{1}{2}t^2}\,dt - (\eta^2+1)\,\eta\, e^{-\frac{1}{2}\eta^2}\right],\quad 0\right\}.$$

Figure 1 shows a set of streamlines for the section of the vortex ring by the half plane $\phi = 0$ calculated from the first approximation $f_1(\eta, \theta)$ given by (23) and drawn at equal increments of the stream function. The vortex ring character of the flow is brought out clearly, but the pattern differs from that of an ordinary vortex ring without buoyancy (cf. Lamb, 1932, p. 238) in that there is a larger core to the ring. This is to be expected on account of the inherently stable nature of such a light core under the forces produced by rotation about its axis.

The second approximation to the temperature field. This approximation will determine the temperature field up to the term in A, and is to be compared with the approximation to the flow field obtained above. The excess temperature variable $h_1(\eta,\theta)$ satisfies the differential equation (16*b*)

$$-\left(3h_1+\eta\frac{\partial h_1}{\partial\eta}\right)+\frac{1}{\eta^2\sin\theta}\frac{\partial(h_0,f_1)}{\partial(\eta,\theta)}=\left(\frac{1}{\eta^2}\frac{\partial}{\partial\eta}\eta^2\frac{\partial}{\partial\eta}+\frac{1}{\eta^2\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}\right)h_1,$$

and if the solutions for h_0 and f_1 are substituted this reduces to

$$\left\{\frac{\partial^2}{\partial\eta^2}+\left(\eta+\frac{2}{\eta}\right)\frac{\partial}{\partial\eta}+\frac{1}{\eta^2}\left(\frac{\partial^2}{\partial\theta^2}+\cot\theta\frac{\partial}{\partial\theta}\right)+3\right\}h_1$$
$$=-\frac{1}{8\pi^3\eta^2}e^{-\frac{1}{2}\eta^2}\left\{\int_0^\eta e^{-\frac{1}{2}t^2}dt-\eta\,e^{-\frac{1}{2}\eta^2}\right\}\cos\theta.$$

The independent variables can be separated if it is supposed that h_1 is of form $h_1(\eta,\theta)=x(\eta)\cos\theta$, where $x(\eta)$ satisfies the ordinary differential equation

$$\frac{d^2x}{d\eta^2}+\left(\eta+\frac{2}{\eta}\right)\frac{dx}{d\eta}+\left(3-\frac{2}{\eta^2}\right)x=-\frac{1}{8\pi^3\eta^2}e^{-\frac{1}{2}\eta^2}\left\{\int_0^\eta e^{-\frac{1}{2}t^2}dt-\eta\,e^{-\frac{1}{2}\eta^2}\right\}. \tag{24}$$

The general solution of equation (24) can be written

$$x(\eta)=-\frac{1}{8\pi^3}\frac{1+\eta^2}{\eta^2}e^{-\frac{1}{2}\eta^2}\int_0^\eta\frac{t^2e^{\frac{1}{2}t^2}}{(1+t^2)^2}\int_0^t\left\{\frac{1+s^2}{s^2}e^{-\frac{1}{2}s^2}\int_0^s e^{-\frac{1}{2}\alpha^2}d\alpha-\frac{1+s^2}{s}e^{-s^2}\right\}ds\,dt$$
$$+c_5\frac{1+\eta^2}{\eta^2}e^{-\frac{1}{2}\eta^2}\int_0^\eta\frac{t^2}{(1+t^2)^2}e^{\frac{1}{2}t^2}dt+c_6\frac{1+\eta^2}{\eta^2}e^{-\frac{1}{2}\eta^2}$$
$$=\frac{1}{32\pi^3\eta^2}e^{-\frac{1}{2}\eta^2}\{\eta\,e^{-\frac{1}{2}\eta^2}-\sqrt{(\tfrac{1}{2}\pi)}\,(1-\eta^2)\operatorname{erf}(\eta/\sqrt{2})\}$$
$$+\left(c_5-\frac{1}{16\pi^3}\right)\frac{1+\eta^2}{\eta^2}e^{-\frac{1}{2}\eta^2}\int_0^\eta\frac{t^2e^{\frac{1}{2}t^2}}{(1+t^2)^2}dt+c_6\frac{1+\eta^2}{\eta^2}e^{-\frac{1}{2}\eta^2}, \tag{25}$$

where $\sqrt{(\frac{1}{2}\pi)}\operatorname{erf}(t/\sqrt{2})=\int_0^t e^{-\frac{1}{2}\alpha^2}d\alpha.$

The boundary conditions on h_1, that is on $x\cos\theta$, are that $x(\eta)\cos\theta\to 0$ as $\eta\to\infty$, $\eta^{-1}x(\eta)\sin\theta=0$ on $\theta=0,\pi$ for all η including $\eta=0$, and the integral condition

$$\int_0^\infty\int_0^\pi\eta^2x(\eta)\sin\theta\cos\theta\,d\theta\,d\eta=0.$$

For small values of η $\qquad x(\eta)=c_6/\eta^2+\frac{1}{2}c_6+O(\eta),$

and hence $c_6=0$ since $x(0)=0$. The value of $c_5-(16\pi^3)^{-1}$ can be found using the integral condition, and the fact that this integral vanishes as the difference of two equal (but not large) parts; thus in spite of the fact that the integration over θ is zero, the integration over η must also be finite and in consequence $c_5-(16\pi^3)^{-1}=0$. Thus the excess temperature perturbation is given by

$$h_1(\eta,\theta)=\frac{1}{32\pi^3\eta^2}e^{-\frac{1}{2}\eta^2}\{\eta\,e^{-\frac{1}{2}\eta^2}-\sqrt{(\tfrac{1}{2}\pi)}\,(1-\eta^2)\operatorname{erf}(\eta/\sqrt{2})\}\cos\theta, \tag{26}$$

and the whole temperature field to the present approximation is

$$T-T_0 = \frac{\kappa\nu A}{8\beta g(\pi\kappa t)^{\frac{3}{2}}}\,e^{-\frac{1}{2}\eta^2}\left\{1+\frac{A\cos\theta}{4(2\pi)^{\frac{1}{2}}}\left(\frac{1}{\eta}e^{-\frac{1}{2}\eta^2}-\sqrt{\frac{\pi}{2}}\,\frac{1-\eta^2}{\eta^2}\,\mathrm{erf}\,\frac{\eta}{\sqrt{2}}\right)\right\}. \tag{27}$$

This is symmetrical about the vertical ($\theta = 0, \pi$) through the origin, and the effect of the term (26) just calculated is to provide a small increase in temperature above the horizontal plane through the origin and an antisymmetrical decrease in temperature below this plane. The modification is small, and at the point where h_1 has its greatest value amounts to roughly $A/100$ times the magnitude of the initial term (relation (14)). The pattern of this modification due to the term $h_1(\eta, \theta)$ is shown by itself in figure 3 for the upper half of the thermal vortex; for

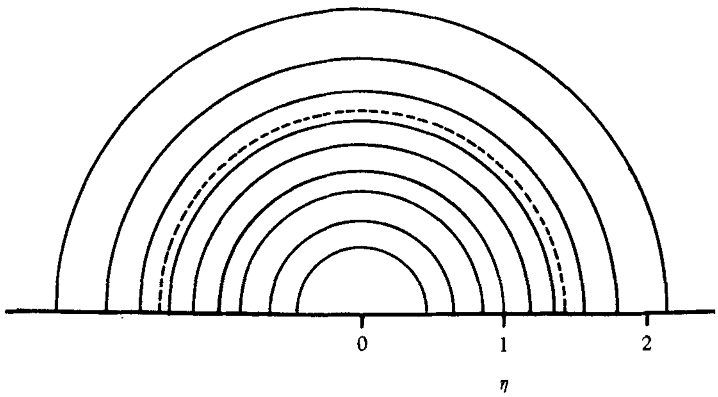

FIGURE 2. The basic approximation to the temperature field for a weak thermal vortex ring. This is the distribution produced in a solid by the instantaneous release of heat from a point. Curves are shown for equal increments of temperature (in tenths of the temperature at the centre); at the broken curve the temperature is $1/e$ of the central value, and this may be used to compare the scale of the temperature and velocity field.

purposes of comparison and so that the scale of the temperature distribution may be compared with that of motion (shown in figure 1) the first approximation to the temperature field (expression (14)) is illustrated in figure 2, again for the upper half of the thermal vortex only. Although there is a maximum value of h_1 at a distance roughly $\eta = 1$ above the origin, because of the relative smallness of this term the point of maximum temperature in the fluid will be much nearer the origin.

The point of the vortex ring at which the temperature is greatest will be vertically above the origin of release at a height which can be found by putting $\partial T/\partial r = 0$ from expression (27), and which is given by

$$\eta = \frac{A}{6(2\pi)^{\frac{1}{2}}} - \frac{13}{10}\left(\frac{A}{6(2\pi)^{\frac{1}{2}}}\right)^3 + \dots.$$

Thus to the order of the present approximation $\eta = A/[6(2\pi)^{\frac{1}{2}}]$.

It may be noted that the distribution of heat in the thermal vortex ring is basically similar to the pattern of thermal diffusion from an instantaneous point release of heat in a solid, with the modification produced by the term $h_1(\eta, \theta)$

shown in figure 3 and resulting in a displacement upwards of the heat relative to the flow field, so that temperatures are increased in the upper parts and decreased in the lower parts of the vortex ring. There is no concentration of heat into the core of a weak thermal vortex ring.

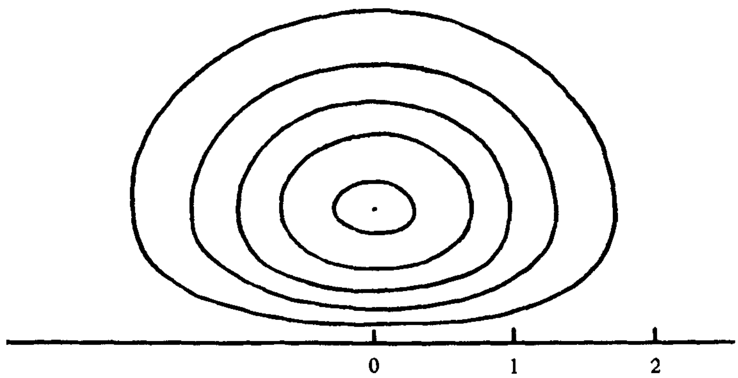

FIGURE 3. The next approximation (h_1) to the temperature field plotted separately for the upper half of the thermal vortex; the contours shown are of equal temperature increase, while a similar set of contours below the dividing horizontal will correspond with lines of equal temperature decrease relative to the basic approximation shown in figure 2. Maximum temperatures in this field are small relative to those at corresponding positions of figure 2. There is an increase of temperature corresponding to the upper half plane and a decrease for the lower.

Behaviour of weak thermal vortex rings

The next approximation to the flow field will introduce a small asymmetry which will have the effect of raising the level of the core axis above that of the origin in the similarity profile, but apart from this will introduce relatively little change in the flow pattern. Thus it will be sufficient to regard the 'centre' of the thermal vortex ring as the point at which the fluid temperature is a maximum, and this point can be identified with the information now available. According to this viewpoint, the actual height h of the vortex ring above its point of release at time t is approximately

$$h = \frac{A}{12\pi}\sqrt{\frac{\kappa t}{\pi}};$$

and the vertical velocity V of the vortex ring is

$$V = \frac{A}{24\pi}\sqrt{\frac{\kappa}{\pi t}}.$$

In addition, the radius R of the vortex ring measured from the vertical axis to the centre of the core is

$$R = 1{\cdot}512\sqrt{(2\kappa t)},$$

and the circulation K is

$$K = \frac{\kappa A}{4\pi}.$$

The energy of the system is

$$T = \frac{15}{64\pi}\rho\kappa^2 A^2 \sqrt{\frac{\kappa t}{2\pi}},$$

and the impulse is wholly upwards with magnitude

$$P = \frac{5}{4\pi}\rho\kappa^2 t A.$$

It should be noted that these have been calculated from the solution for $\sigma = 1$ (i.e. $\nu = \kappa$); they should give a good approximation for values of σ not too great or too small relative to unity, but in this case κ in the results above should be replaced by $(\kappa\nu)^{\frac{1}{2}}$ to take account of the equal importance of thermal conduction and viscous diffusion in establishing the character of convection.

It is possible now to compare these results with those of Turner (1957), although it must be borne in mind that Turner considered relatively strong vortex rings in which much of the buoyancy is concentrated into a ring core, whereas in these weak thermal vortices there is no concentration of the heat into a ring. Indeed the special features of these weak vortices is that they combine a flow field which is very much the same as that of a normal vortex ring, with a completely different distribution of temperature with a central maximum and temperature decreasing radially outwards in all directions and with a small asymmetry about the horizontal. The diameter of the ring increases linearly with height according as

$$AR = 45{\cdot}5h;$$

hence the rate of spread of a weak thermal vortex ring in terms of its rate of propagation is very much greater than for a normal buoyant vortex ring (for which $R \doteqdot 0{\cdot}2h$, say, from Turner). In fact the very weakest vortices scarcely propagate at all, although they increase in size at the same rate (depending only on $\kappa\nu$) as stronger ones. This brings out clearly the fact that the growth of weak vortices is due essentially to molecular diffusion (thermal and viscous) and not to the more familiar process of mixture with the environment produced by drawing ambient fluid into the rear of the ring. Moreover, this explains the differences found in the temperature field, for any ambient fluid which is drawn into the centre of a weak vortex enters so slowly that on the way it is heated by conduction to very nearly the previous temperature at the centre.

It may be noted that the circulation K remains constant and is proportional to the Rayleigh number A; thus the flow in the vortex ring is started impulsively at the initial instant, though the ring itself moves off from rest. The impulse, $P = 1{\cdot}094\rho K R^2$, is proportionally smaller than the value taken by Turner on account of the lower rate of propagation of weak vortices. The value of Turner's constant $c = VR/K$ is in this case $0{\cdot}20$; this is surprisingly close to his experimental value $0{\cdot}27$ found for strong buoyant rings of orthodox pattern, because the decrease in velocity is offset by the increase in radius, and it may well be that this constant varies little over a very wide range of vortex strengths.

The arguments for stability of the normal vortex ring no longer apply when the heat is concentrated towards the centre and not into the ring, and there is no reason why weak thermal vortices should be particularly stable to disturbances. If the flow becomes unsteady there will be increased mixing with the ambient fluid in the outer sheaths and fluid drawn back towards the centre will be colder. This is probably the way in which orthodox vortex rings are produced.

As far as can be estimated from the solution to the stage it has been calculated, the results are likely to give a reasonably good picture of thermal vortices up to Rayleigh numbers of 10 or more, and in the absence of stability effects this behaviour is unlikely to change radically up to Rayleigh numbers of a few hundred, say. However, even these correspond to a release of heat in air of the order of a thousandth of a calorie, or in water of a calorie. And so the solution given above is of very little practical use, although it is interesting because it shows the existence of a different temperature distribution in weak thermal vortex rings, and because the solution in closed form can be used for a number of calculations.

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